Reprinted from JOURNAL OF COMBINATORIAL THEORY, Series B
All Rights Reserved by Academic Press, New York and London

Vol. 37, No. 1, August 1984
Printed in Belgium

Note

Every 4-Regular Graph Plus an Edge Contains a 3-Regular Subgraph

N. ALON*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts

S. FRIEDLAND

Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel

AND

G. KALAI*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts

Communicated by the Managing Editors

Received July 25, 1983

Let $G=(V,E)$ be a 4-regular loopless graph plus an edge with $|V|=n$ vertices and $|E|=m=2n+1$ edges. (G may contain multiple edges.) Let $a_j^{(i)}$ be the (j,i)th entry of its (vertex, edge)-incidence matrix. As shown below, Chevalley's classical theorem implies that there exists $\phi \neq I \subseteq \{1,2,\ldots,m\}$ such that

$$\sum \{a_j^{(i)} : i \in I\} \equiv 0 \pmod 3, \qquad (j=1,2,\ldots,n). \tag{1}$$

Hence G contains a 3-regular subgraph. Note that a graph on 3 vertices with 2 parallel edges between any two shows that the "plus an edge" cannot be omitted.

* Research supported in part by the Weizmann Fellowship for Scientific Research.

0095-8956/84 $3.00

CHEVALLEY'S THEOREM. [3] *For* $j = 1, 2, \ldots, n$ *let* $F_j(x_1, \ldots, x_m)$ *be a polynomial of degree* r_j. *Suppose* $\sum_{j=1}^{n} r_j < m$. *If p is a prime and the system of congruences*

$$F_j(x_1, x_2, \ldots, x_m) \equiv 0 \pmod{p} \qquad (j = 1, 2, \ldots, n)$$

has one solution then it has at least two solutions.

Proof of (1). Consider the system of congruences

$$\sum_{i=1}^{m} a_j^{(i)} x_i^2 \equiv 0 \pmod{3} \qquad (j = 1, 2, \ldots, n).$$

Since $2n < m$ and $\mathbf{0}$ is a trivial solution, Chevalley's Theorem implies the existence of a nontrivial solution $\mathbf{x}$. Let I be the set of indices of its nonzero coordinates. Then $\phi \neq I \subseteq \{1, 2, \ldots, m\}$ and if $i \in I$, $x_i^2 \equiv 1 \pmod{3}$. Hence (1) holds as required.

Remark. The result mentioned in the title is related to the well known Berge–Sauer conjecture [2], which has recently been proved [4]. In [1] we apply Chevalley-type theorems to prove more general graph theoretical results. Since the proofs in [1] are somewhat complicated, while the basic idea is very simple, we follow the referee's suggestion and publish this communication separately.

REFERENCES

1. N. ALON, S. FRIEDLAND, AND G. KALAI, Regular subgraphs of almost regular graphs, *J. Combin. Theory Ser. B* **37** (1984), 79–91.
2. J. A. BONDY AND U. S. R. MURTY, "Graph Theory with Applications," p. 246, Macmillan & Co., London, 1976.
3. Z. I. BOREVICH AND I. R. SHAFAREVICH, "Number Theory," Chap. 1, Academic Press, New York, 1966.
4. V. A. TAŠKINOV, Regular subgraphs of regular graphs, *Soviet Math. Dokl.* **26** (1982), 37–38.

Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium